Algorithms, Algebra, and Access

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Introduction

Perhaps the most contentious dispute arising from recent efforts to reform mathematics education concerns the place of traditional arithmetic algorithms in the elementary school curriculum. Opposition to teaching algorithms, especially the one for long division, can be traced at least as far back as the National Council of Teachers of Mathematics (NCTM) 1989 Standards Document's call for de-emphasis of formal manipulation skills [N]. Indeed, prominent educators, including Steve Leinwand and Constance Kamii, have proposed that the standard algorithms be banished from American classrooms because they undermine children's understanding, performance, and even emotional well-being [K] [L]. None of these effects has ever been documented by a properly controlled experimental study using a statistically valid sample population.

Most new curricula of the past decade were based on the 1989 NCTM Standards and did in fact de-emphasize the standard algorithms or omit them entirely. Although the year 2000 revision of the Standards advocated "investigating" some of those algorithms as one of several possible techniques of whole number calculation [N1], the practical effect of that apparent revision of perspective remains to be seen.

This paper will argue unequivocally for the critical role played by the standard algorithms of arithmetic in students' mathematical development and in their preparation for mathematics-based careers. To do so, it will address misunderstandings supporting the view that student-invented procedures and other alternative procedures provide a mathematically valid substitute for, rather than a pedagogically useful supplement to, traditional methods for whole number calculation [K][L][S]. Its recommendation can be stated succinctly as follows. Instead of being forced into programs that de-emphasize, denigrate, or discard the traditional algorithms of arithmetic, all American children should receive balanced K-8 mathematics instruction that includes appropriate emphasis on the development of formal and algorithmic skills. Only then will they have a meaningful chance to develop their own competencies and thereby obtain access to careers that are both personally rewarding and crucial to the well-being of the larger society.

The importance of formal algorithmic skills to college-bound students was underscored in the May, 2001 report of a commission of experts (the "Math Commission"), chaired by City University of New York Chancellor and mathematician Matthew Goldstein, and charged by Schools Chancellor Harold M. Levy with reviewing mathematics education in the New York City school system [Go]. Recommendation 2 of that report calls for establishing

> "a new option for all interested students in grades 9-11, emphasizing formal and abstract mathematical competency."

"Formal and abstract mathematical competency," henceforth abbreviated FAMC, denotes, among other things, the ability to work efficiently and accurately with the symbolic algebraic manipulations emphasized in the traditional high school curriculum. A principal theme of this essay is that children's early exposure to, and adequate practice with, the standard algorithms of arithmetic constitute critical first steps in acquiring FAMC.

Concerning the recommended new option, the following remarks, included in the Math Commission's publicly distributed draft document but omitted from its final report, should be of particular interest to curriculum developers and to faculties of schools of education:

> "It is important to recognize that the goals of the current New York State curriculum…come at a cost. Whenever an emphasis is placed on ensuring that applications are made to 'real world' situations …less emphasis is placed on arithmetical or mathematical ideas and the formal, abstract contextual settings sought particularly by students who will go on to become scientists, engineers, mathematicians, computer scientists, physicians, and educators of mathematics.
>
> "Despite their many strengths, *the NCTM standards do not contain the rigor, algorithmic approach, formal methods, and logical reasoning which are required of this small but critically important portion of the population."* (emphasis added)

It is not without significance for the mathematics education community that "educators of mathematics" are included in the list of professionals who require training in an algorithmic approach and formal methods. That list is not complete, of course: many programs in nursing, physical therapy, occupational therapy, psychology, architecture, economics, business, and other professions require students to complete or to have completed courses in calculus, statistics, or physics.

The draft document's quoted characterization of the portion of the population requiring FAMC as "small" is misleading. In the Fall, 2000 semester, when 1.156 million U.S. high school graduates began a four-year college, 463,000 students enrolled in first-year calculus. In addition, there were 236,000 enrollments in undergraduate statistics courses [BLS][LMR]. In fact, therefore, the cohort of students aiming for careers that require calculus and statistics is quite large. Furthermore, that group can only be expected to increase in size and importance as computer and other technologies advance. Students who graduate high school with inadequate FAMC will be unable to succeed in college mathematics, and as a result will confront circumscribed career and life choices.

The Math Commission report does not address a critical question: what type of K-8 curriculum will, or will not, adequately prepare students for success in the proposed FAMC high school mathematics option? A principal motivation for writing this essay is concern that students whose K-8 mathematics programs de-emphasize or eliminate traditional algorithmic approaches will be effectively denied access to that option, or indeed to any high school program designed to prepare them for rigorous college mathematics. Among the groups most severely impacted will be the children of non-English speaking immigrants, children who until now have been able to enter the mainstream of American society by pursuing careers that emphasize mathematical, as opposed to linguistic, competency.

This paper is organized as follows. Section 1 lists “myths” and “facts” about standard, alternative, and student-invented algorithms. A gentler choice of words would be "assertions" and "discussions." The more eristic terminology is used because it appears to have become standard usage in discussions of educational issues.

Section 2 analyzes a well-known alternative division algorithm, described and advocated in a paper by Judith Sowder, a professor of math education at San Diego State University [S]. That procedure is shown to be cumbersome and error-prone when applied to a representative problem, rather than to the single unusually easy demonstration problem Sowder uses in an attempt to portray the alternative algorithm as reasonably efficient.

Section 3 describes how and why the traditional algorithms of arithmetic are indispensable preparation for the study of algebra with variables, and therefore also for physics, statistics, and calculus courses required by students who must acquire FAMC.

Finally, an appendix describes and illustrates the standard algorithms for multiplication and division of multi-digit decimal integers. It is hoped that the explanations therein will convey both the ideas underlying the algorithms and their formal implementation in a style that can be expanded and shared with students at all levels, up to and including prospective and in-service elementary and middle school teachers.

1. Myths and Facts

This section will refute a number of assertions that are often marshaled to argue against the continued study of the standard algorithms, each based on specific misunderstandings that will be addressed in detail.

Myth 1: It is no longer sensible for children to learn the standard algorithms because the right answer can be found much more easily with a calculator.

Fact: The importance of teaching algorithms was addressed in the 1999 open letter addressed to then Secretary of Education Richard Riley and signed by 200 mathematicians and scientists to protest the designation of 10 NCTM-based programs as "promising" or "exemplary"[OL]. To refute the extreme view that "continuing to teach [pencil-and-paper algorithms] to our students is not only unnecessary, but counterproductive and downright dangerous [L]," the letter stated:

> In sharp contrast, a committee of the American Mathematical Society (AMS) ….published a report which stressed the mathematical significance of the arithmetic algorithms, as well as addressing other mathematical issues, [including] the statement:
>
> "*We would like to emphasize that the standard algorithms of arithmetic are more than just 'ways to get the answer' -- that is, they have theoretical as well as practical significance. For one thing, all the algorithms of arithmetic are preparatory for algebra, since there are (again, not by accident, but by virtue of the construction of the decimal system) strong analogies between arithmetic of ordinary numbers and arithmetic of polynomials.*"

What is meant, for instance, is that the polynomial multiplication problem (2x+3y)(4x+5y) is an algebraic generalization of the arithmetic problem 23*45. If (and this is a crucial qualification) the multiplication algorithm for whole numbers has been taught and learned intelligently, and if the formal manipulation component of that algorithm has been mastered, the formal and conceptual jumps from numbers to polynomials will be manageable. A student who sees the algebra problem for the first time will think or say "I've been here before," especially if her algebra teacher is aware of and can communicate effectively the analogy between the arithmetic and algebraic versions of the multiplication algorithm. Conversely, a student who encounters the polynomial multiplication problem without having seen and practiced the whole number algorithm will be at a severe, if not crippling, disadvantage.

More generally, it is naïve and counterproductive to suggest that calculators, or even symbolic algebra software, eliminate or reduce students' need for formal algebraic skills. If technology is introduced appropriately, as a tool for obtaining numerical and graphical solutions to problems whose algebraic representations are too complex for hand calculation, facility with algebraic manipulation is crucial both for building those representations and for correctly inputting them into the computing device. In my own experience with technology-based courses, it is precisely those students with the weakest symbolic and algebraic skills who have the most trouble correctly entering formulas and expressions into their calculator or computer.

Myth 2: Children who use the standard algorithms to solve sheets of drill problems are not thinking about what they are doing and thus are engaging in robot-like behavior that stifles their development as learners of mathematics.

Fact: This myth, stated in a larger context, was deflated nearly a century ago by the eminent mathematician and philosopher Alfred North Whitehead in his *Introduction to Mathematics* [W]:

It is a profoundly erroneous truism repeated by all copybooks, and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of operations that we can perform without thinking about them. Operations of thought are like cavalry charges in a battle--they are strictly limited in number, they require fresh horses, and must only be made at decisive moments.

What holds true of civilization is all the more true of mathematics. It is sensible to encourage children to think about all the details of the mathematics problems they are solving if, and only if, the *final* goal of their study of mathematics is to perform simple tasks such as adding or multiplying one- or two-digit numbers. For students to progress beyond this basic level, however, it is neither possible nor desirable to demand that they constantly think about the details of the steps and sub-steps of even moderately complicated problems.

Any student taking a first year calculus exam, for example, must perform hundreds of small operations automatically and accurately. Indeed, a fundamental difficulty that bedevils many calculus students is that they have not learned to perform lower level mathematical operations automatically, accurately, and, as Whitehead advises, without thinking about what they are doing. Only by submerging a concern with irrelevant detail can students choose, develop, and execute an appropriate global strategy for solving a complicated problem.

How do students acquire the ability to perform lower level operations automatically? Numerical and algebraic symbol manipulation skills are not inborn. They must be learned, and for most students the process is not easy. Children need to begin slowly, with a few carefully chosen examples, in order to gain an understanding of how an algebraic process works. After that initial stage, practice for the sake of practice, i.e. drill for skill, is the path whereby the vast majority of students can reach the level of fluency and accuracy that is needed for formal mathematical competency.

Myth 3: Children can invent their own algorithms.

Fact: A mathematical algorithm (hereafter referred to simply as "an algorithm") is an ordered sequence of steps that yields the solutions to a well-defined class of problems. For example, one could devise algorithms for any of the following classes of problems involving whole number operands:

Multiply a 2-digit number by a 1-digit number.
Divide any number by a 1-digit number.
Multiply any two numbers.

The algorithm must be formulated as a sequence of mathematical or logical steps in a way that describes exactly how to find the solution to every problem in the class, rather than by just demonstrating the procedure for a few specific examples.

Any algorithm that purports to solve a class of problems must be both *reliable* and *efficient.* Reliable means: absent careless errors, a person who follows the steps in the algorithm will obtain the correct answer to every problem in the given class. Efficient means: on average, or perhaps always, the algorithm uses as few arithmetic steps as possible.

A procedure that solves some but not all problems in a given class is not an algorithm, but is rather a trial and error method that is certainly not reliable and is usually not efficient. While efficiency need not be the highest virtue for pedagogical purposes, reliability must not be compromised, for there is no *a priori* way to determine whether an unreliable algorithm can be used to solve a given problem. Throughout history, it has been a challenging task for adult mathematicians to devise reliable and efficient algorithms that apply to general classes of operands. Any claim that today's K-8 students can do the same should be subjected to the most intense scrutiny.

In view of such skepticism, the reader may wonder how it can be that so many curricula seem to overflow with examples of clever student solutions. Part of the explanation is that the authors carefully select problems that are well suited to *ad hoc* methods and then avoid analyzing either the reliability or the efficiency of the solution strategy. The following typical example, from the TERC teacher's manual for the Investigations unit *Building on Numbers You Know,* shows how a student calculated 374 divided by 12 [T].

12 times 10 = 120
12 times 20 = 120 + 120 = 240
12 times 30 = 240 + 120 = 360
12 times 31 = 360 + 12 = 372 and so the quotient is 31 with remainder 2.

This strategy does yield a pleasantly compact solution –for the chosen three-digit dividend and two-digit divisor. How well does a similar approach work with another problem of the same type, such as 967 divided by 18?

10 18´s are 180
20 18´s are 180 + 180 = 360
40 18´s are 360 + 360 = 720
50 18´s are 720 + 180 = 900
Subtract 900 from the dividend 967, leaving 67
3 20's are 60 and so
3 18´s are 54 (or, perhaps, 18 + 18 = 36 and so 36 + 18 = 54)
Subtract 54 from 67, leaving 13, and so the quotient is
50 + 3 = 53 and the remainder is 13

Applied to this second problem, the TERC strategy appears rather clumsy. Indeed, it is quite inefficient when compared with the traditional algorithm, which replaces the first four TERC steps by the observation that 18 goes into 96 (tens) 5 times.

Which of the two examples above more fairly represents the complexity and efficiency of the TERC solution strategy? It's hard to say, because that strategy is not well defined. However, the basic reason for the clean solution in the TERC example is apparent: the quotient is

31 = 10 + 10 + 10 +1, the sum of only four summands.

In contrast to the TERC procedure, an alternative division algorithm explained and advocated by Judith Sowder in [S] is both well defined and reliable. Curiously, she purports to demonstrate the efficiency of the variant algorithm by using a single example, for which the quotient also turns out to be

31 = 10 + 10 + 10 + 1, the sum of the same four summands!

The analysis in Section 2 will demonstrate quite clearly the inefficiency of Sowder's algorithm when it is applied to randomly chosen problems rather than to this carefully selected, very easy, and apparently rather popular demonstration example.

Myth 4: The traditional long division algorithm is obscure and hard to understand for both students and teachers.

Fact: Here is an attempt at a clear exposition. Let's start with a simple example and define some standard terms.

In the problem 63 divided by 10,
63 is the dividend;
10 is the divisor;
6 is the (integer part of the) quotient; and
3 is the remainder.

To find the quotient, observe that 6 times 10 is the highest multiple of 10 that is less than 63, and so, for example, dividing 63 children into groups of 10 children each would separate them into 6 groups and leave 3 children "left over." Therefore the quotient is 6 with a remainder of 3.

The standard division algorithm extends this method to more difficult problems with multi-digit dividends and divisors. It works as follows. You need to keep track of two numbers, the "current dividend" and the "current quotient," each of which will be updated one or more times during the execution of the algorithm. To begin, set the current dividend equal to the dividend, and set the current quotient equal to 0. To continue, perform the following steps until Step 1 tells you to stop.

Step 1: If the current dividend is less than the divisor, stop: the quotient is the current quotient and the remainder is the current dividend. Otherwise, continue with

Step 2: From among the multiples of the divisor by these multipliers:
1, 2, 3, 4, 5, 6, 7, 8, 9,
10, 20, 30, 40, 50, 60, 70, 80, 90,
100, 200, 300, 400, 500, 600, 700, 800, 900, , ,

- Select the largest multiple that is less than or equal to the current dividend.

- Subtract that multiple from the current dividend to update the current dividend.
- Add the current quotient and the selected multiplier to update the current quotient.
- Go to Step 1.

The above description is complete, albeit condensed, but does not prescribe a method for writing down the solutions to particular problems. There are many possible tableaux (or formats) for doing so, each illustrating a practical method for implementing the long division algorithm. The appendix to this paper contains an expanded explanation of that algorithm, together with both the tableau used in the United States and a variant that may help students and teachers to understand better the underlying ideas.

Myth 5: There are many long division algorithms. In fact, students in other countries don't learn the standard algorithm.

Fact: There are only a handful of reliable methods for whole number division in the decimal number system. Of these, the standard algorithm is by far the most efficient. The following section will quantify just how much more efficient it is than one frequently discussed variant.

The standard algorithm's superior efficiency makes its use universal, or at least planet-wide. What is not completely uniform, however, is the tableau that is used. There are minor variations from country to country in both the number and positioning of the intermediate calculations that are written down by the problem solver. However, any country's tableau translates easily to any other country's. A student who has learned long division in a Russian elementary school, for example, should be able to explain the correspondence between his tableau and its American counterpart by seeing how Steps 1 and 2 of the division algorithm specified above are carried out in each. The variations are notational rather than procedural and have little if any effect on students' understanding or on the amount of time they need to solve problems.

Since the various tableaux now used in different countries differ only in minor details, it is appropriate to refer to a "modern" tableau for long division: modern, but not recent, for that tableau already appeared in print in the year 1491[C]. Its brevity, comprehensibility, and ease of use have never since been surpassed.

Before that time, the standard algorithm was also the basis for division calculations, but the tableau used was clumsier and more prone to error than the modern one. Although part of the difference in ease of use stemmed from the fact that the earlier "galley" tableau had been designed for use with a sand table (a device that facilitates frequent crossing out and rewriting of digits), the major improvement offered by the modern tableau was that it made transparent the underlying logic of the division algorithm specified above in Fact 4. See the appendix for details. For further discussion of various tableaux for division and other algorithms, as well as for a general introduction to the history and importance of the algorithms of arithmetic, a good reference is Frank Swetz' engrossing exposition *Capitalism and Arithmetic* [Sw][1].

In recent years, a muddling of the distinction between the terms "algorithm" and "tableau" has led to frequent miscommunication between mathematicians and math educators. Both methods books and student texts typically explain how to do long division not by specifying an algorithm as defined earlier, but rather by illustrating a few specific problem solutions using the tableau of choice. Such presentations conceal the underlying logic of the standard algorithm and all too easily descend into lists of mechanical instructions, a mode of pedagogy justifiably criticized in the anti-algorithm literature. Such criticism should be leveled against the exposition in those texts rather than against the algorithms themselves. Both an explicit description of the division algorithm *and* clear demonstration examples worked with a given tableau, as provided in Fact 4 above and in the appendix to this paper, are prerequisites for achieving proficiency with the use of the standard algorithm as well as for acquiring a reasonable sense of how and why it works.

Before proceeding, it is important for the reader to understand what is, and what is not, being advocated in this paper. The essential message is: as one component of their elementary education, students need to acquire an understanding of, and automatic proficiency with, the standard algorithms of arithmetic. However, it is certainly not being suggested that those algorithms are now taught appropriately, or ever have been taught appropriately, in American schools. Clearly, it is possible to teach algorithms badly. However, if such teaching is prevalent in this country, as it appears to be, the solution is to improve teachers' own understanding of algorithmic methods rather than to cast aspersions on the subject matter.

The need for such improvement became apparent during an examination of a college library's extensive mathematics education collection: it was impossible to locate a coherent specification of the long division algorithm. For this unfortunate situation, it is difficult to suggest a plausible explanation. In an attempt to fill the void, Section 1 and the appendix to this paper contain a detailed specification and overview, with examples, of both the multiplication and long division algorithms.

Similarly, it is no secret that exercises geared toward the development of formal mathematical skills are not the high point of students' educational experience. Much can be done to increase students' interest in and enthusiasm for genuine mathematical activity. In particular, there is a great need for curricula that provide engaging and open-ended computational activities, examples that illuminate rather than obscure the power of symbolic representations, and mathematically honest but age-appropriate discussions of the ideas underlying the decimal number system[2]. For examples worth emulating, a good

[1] Note however, that Swetz' assertion on p. 217: "The prolonged historical popularity of the galley method of division was due in great part to its efficiency and resulting economy" is extremely misleading insofar as it suggests a favorable comparison with the modern tableau [O].
[2] For example, given any collection of marbles, why is there one, and only one, decimal whole number that tells you the size of the collection?

source is *Math Workshop* [WBSB], an innovative K-6 curriculum developed during the Sputnik era by an unusually diverse and capable group of mathematicians and educators.

Finally, nothing in the following discussion should be construed as criticism of having students develop their own strategies for attacking arithmetic problems. Experimenting with numbers is an important part of children's mathematical development. They certainly ought to know that a number of everyday calculations, such as finding the price of 7 bottles of soda costing 98 cents each, are best approached using a strategy other than the standard algorithm. However, both students and educators also need to understand that strategies for particular types of problems do not generalize to algorithms for general classes of problems, and furthermore that only the standard algorithms of arithmetic adequately model the particularly troublesome polynomial multiplication and division algorithms that are critical for the study of algebra and calculus.

2. An alternative long division algorithm?

To appreciate the background for some of the myths of the previous section, it is well worth analyzing part of an article by Judith Sowder[S]. Therein she claims to describe an algorithm for long division that is "easier" and "better remembered" than the standard one. Here is the relevant portion of the article, indented and in a smaller font, with my observations interspersed.

> How many of you can recall learning the long division algorithm? Suppose I tell you that I can show you an easier way to do long division, albeit a way that may take slightly more time to do. (Of course, if you weigh the time spent learning the method that does not make sense to you, perhaps time favors my procedure.) Here is a situation: As parents of the new septuplets, you estimate that you need 12 diapers per baby per day, or 84 diapers per day. You have been given 2664 diapers. How many days will these diapers last? There are, of course, different ways of solving this problem, but suppose you decide you are going to solve the problem by dividing 2664 by 84. This is equivalent to asking: How many 84s are there in 2664? We know there are at least ten 84s in 2664 because ten 84s is 840. We know there are not a hundred 84s since that would be 8400, which is more than 2664. So we could begin by removing (that is, subtracting) ten 84s. We'll keep a tally on the right of the number of 84s removed. We see that we can do this again, and even again. But when we have only 144 diapers left, we cannot remove ten 84's again. We can, though remove one more 84. Finally, you have 60 diapers left, not enough for another day. Add up the tallies, and we see that the diapers will last 31 days, and we will have 60 diapers left.

```
2664
-840      10
----
1824
-840      10
----
 984
-840      10
----
 144
- 84       1
----      --
  60      31
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> If a student had recognized at the beginning that I could take out thirty 84s (2520), then one 84, the student would have the traditional algorithm but for the placement of the digits and the writing down of some extra zeros.

Sowder has modified the standard algorithm by permitting only powers of 10 as multiples of the divisor that may be subtracted from the dividend in Step 2 of the division algorithm described at the end of Section 1. Her method, hereafter called the "alternative division algorithm," is a reasonable way to introduce long division. The effect of the modification, however, is to replace each multiplication step of the standard algorithm by a number of subtractions equal to the new digit in the quotient.

> But for most children, it takes as much time to determine that the first digit in the quotient is 3 as to subtract 840 three times.

It would be enlightening to learn more about the sample population that Sowder studied in order to draw this sweeping inference. If the statement is true, it says more about the way children have been taught mathematics than about their mathematical capabilities. I would hope there is general agreement that finding out how many times 840 goes into 2664 is a sensible example for teaching estimation skills to children. The ability to respond, automatically when asked, that 8 (hundred) goes into 26 (hundred) three times is a skill that every student should acquire before taking a high school mathematics course and most certainly before arriving in college.

> Students can make sense of division this way. It is not quite so efficient as the standard algorithm for long division, but it is less susceptible to error simply because it is understood.

Both assertions should be examined carefully. The second, concerning susceptibility to error, is belied by the significant portion of students worldwide who carry out the algorithm automatically, effortlessly and correctly without bothering to think about how or why it works. Much more important is the claim of reasonable efficiency, which is misleading at best. Consciously or unconsciously, Sowder has chosen an example for which her alternative division algorithm requires just four subtractions, as compared to two subtractions and one multiplication for the standard method. Although the standard algorithm requires writing down only half as many digits as does the alternative algorithm, the assertion that the latter is "not quite so efficient" as the former is not unreasonable. The real question is: has Sowder chosen a representative problem from among all division problems with a four-digit dividend and two-digit divisor? The answer is emphatically negative, as the following analysis demonstrates.

The number of subtractions needed to solve a division problem using the alternative algorithm is the sum of the digits of the integer part of the quotient. For example, a problem with a quotient of 45 requires nine subtractions. In the worst case involving a three-digit quotient, 27 subtractions are required when the quotient is 999.

How many four-digit by two-digit division problems are there? The dividend can range from 1000 to 9999 and so there are 9000 possible dividends. For each dividend, there are 90 possible divisors, since the divisor is between 10 and 99. Therefore the total number of division problems of this type is 90 * 9000 = 810,000. The following table shows percentages of these 810,000 problems grouped according to the number of subtractions required by the alternative algorithm. The results were obtained not by applying analytic methods, but rather by using what many would agree is an appropriate application of technology: writing a fifteen-line computer program that performs the 810,000 division problems and keeps track of the number of subtractions required for each.

Number of Subtractions	Percentage of problems
1 - 4	9.8%
5 - 9	37.4%
10 - 15	43.7%
16 - 27	9.1%

The results are clear.

- Sowder´s example places in the easiest tenth of all problems.
- A problem as atypical as Sowder´s, but in the most difficult tenth, requires 15 or more subtractions.
- A representative 4-digit by 2-digit division problem, namely one at the 50th percentile of difficulty, requires 10 subtractions.

Had Sowder chosen a representative problem, her method would have been revealed as both cumbersome and inefficient. For example, compare the two algorithms for 9265 divided by 27:

```
 9265
-2700  100
 6565
-2700  100
 3865
-2700  100
 1165
- 270   10
  895
- 270   10
  625
- 270   10
  355
- 270   10
   85
-  27    1
   58
-  27    1
   31
-  27    1
    4  343 remainder 4
```

```
   343 remainder 4
27/9265
   81
   116
   108
     85
     81
      4
```

In this typical case, it is apparent that Sowder’s alternative division algorithm does not require just "slightly more" time than the standard one. Her version requires 10 subtractions, of which 7 use “borrowing,” a notoriously error-prone procedure, as well as writing down 79 digits, not counting optional trailing zeroes. In contrast, the standard algorithm requires 3 multiplications and 3 subtractions, only one of which uses “borrowing”, and writing down only 23 digits. Clearly, for typical problems, it is the alternative algorithm that is much less efficient and also much more prone to error. Furthermore, fully twenty per cent of the 810,000 problems done using that algorithm will require considerably more work (14 through 27 subtractions) than the example illustrated, whereas every one of those problems can be solved using the standard algorithm with only 3 multiplications and 3 subtractions.

> Efficiency is no longer a prime consideration, because you yourselves use calculators for long-division problems when they are messy or you are in a hurry. And this procedure will be better remembered, or, if not remembered, reinvented. Contrary to what some people believe, God did NOT hand down on a tablet the standard long-division procedure we all learned. It simply is a streamlined procedure that was invented when computations had to be done quickly, such as in my [earlier] example of the graduate students doing calculations for the scientists [at Los Alamos].

The mathematical and pedagogical (if not the theological) assumptions underlying this remark have already been critiqued in Section 1. It is certainly the case, as Sowder observes, that the standard algorithm is a streamlined procedure and so is well suited to calculation. However, it is simply not true that the only value, or even the principal value, of the standard algorithms of arithmetic is their ability to produce answers. To support this assertion, the following section presents a substantial but only partial list of topics in the high school and college FAMC curricula wherein those algorithms resurface in a more sophisticated form. In particular, it turns out that the alternative division algorithm is not only inefficient at obtaining answers, but is also limited in scope, in that it provides no insight into any of the important algebra topics that are nicely illuminated by the standard algorithm.

3. The importance of teaching algorithms

The underlying logic of the algorithms of arithmetic, if properly taught, illuminates the relationships between the operations of arithmetic in a way that leads naturally to the study of algebra with variables. For example, the polynomial multiplication problem (3x+5)(2x+4) can best be introduced as a generalization of the integer multiplication problem 35*24. Adding polynomials and collecting terms is but a fancier version of adding multi-digit integers while keeping track of place value. The decimal representation of integers provides a model for polynomials. An intelligent motivation of polynomial division problems is impossible if students are ignorant of the standard division algorithm for integers. And so on.

Equally important, practice with arithmetic algorithms is a student’s first experience with the formal manipulation of mathematical symbols. Often lost in educators’ attempts to help children acquire “conceptual understanding” is the following basic reality: formal mathematical competency requires well-developed symbol manipulation skills. Even a glance at the solutions to a standard freshman calculus final exam reveals that students must carry out pages of algebraic manipulations accurately and efficiently. Students who come equipped with symbolic manipulation skills are not guaranteed success in college mathematics. However, those who lack such skills face virtually certain failure in any math or physics course that has not been watered down by the evisceration of algebraic content.

Among the standard algorithms, the long division algorithm is perhaps the most important as preparation for higher mathematical study. Elementary school students deprived of exposure to and practice with that algorithm will be severely handicapped when they encounter applications and generalizations that surface at several stages of their ensuing mathematical education.

The first important application of the long division algorithm can and should be discussed in middle school: rational numbers, and only rational numbers, are expressible as repeating decimals. Indeed, a close look at the modern long division tableau shows that the decimal expansion of the quotient eventually becomes an infinite repetition of a finite sequence of digits, a lovely revelation that is totally obscured by the unwieldy tableau of the alternative division algorithm.

Conversely, any repeating decimal can be written as a rational number by using a fraction whose numerator consists of that repeating sequence and whose denominator consists of as many successive 9's as there are digits in the repeating sequence. Here are two examples in which the repeating sequence of digits is 2,3,4:

- 0.234234234… = 234/999
- $1.345234234... = 1.345 + 0.000234234... = \frac{1345}{1000} + \frac{1}{1000} * \frac{234}{999} = \frac{1343889}{999000}$

If students are willing to believe that infinite decimal expansions make sense (in reality, such expansions must be justified by invoking moderately sophisticated arguments), they can easily discover lots of irrational numbers by devising infinite non-repeating sequences of digits. For example,

$$1.1010010001000010000010000001.....$$

is obtained by concatenating successive powers of ten, whereas

$$1.234567891011121314151617181920 21......$$

is the ordered concatenation of all counting numbers. Proving that these and similar sequences of digits are non-repeating is a useful and tractable exercise in logic for middle school students.

A rather different and very concrete reason for teaching K-8 students the standard long division algorithm is that they will need to know it in order to understand and become fluent with polynomial division, a procedure traditionally taught in high school as a prerequisite for at least three important topics in the college calculus curriculum. Only the standard algorithm for integer division generalizes to the case of polynomials. Indeed, the underlying idea of the alternative algorithm, that division is repeated subtraction, is false in the case of polynomials. The reason is easy to understand. To divide 100 by 10, one can reduce 100 to zero by subtracting 10 from 100 ten times and thereby conclude that the quotient is 10; of course, it would be more efficient to remember that 100 is the product 10*10. However, the only way to find the quotient $x^2 \div x$ is to recall that $x * x = x^2$, a fact that cannot be discovered by using repeated subtraction. Therefore, students who have not already become proficient using the standard algorithm for dividing whole numbers will have great difficulty when their high school algebra instructor explains division of polynomials. They will be totally lost when that algorithm reappears in calculus, for the instructor will assume that students know the traditional algorithm cold.

For a more thorough treatment of the pedagogical and mathematical importance of the long division algorithm, see the essay by David Klein and Jim Milgram [KM].

Conclusion

Referring to her alternative algorithm, Sowder states [S]:

> When I have showed this alternative procedure to prospective teachers, their reaction was sometimes one of anger--anger that they had been made to suffer through the mysterious and difficult long-division algorithm when it really is all so simple to understand.

In such circumstances I too would be unhappy. The real question is: if those prospective teachers are typical, why weren't they taught the long division algorithm by educators who themselves understood the algorithm and its importance for students' later study of algebra? Toward achieving such understanding, it is hoped that the motivated and leisurely exposition of that algorithm in the appendix to this paper will be shared fruitfully by as many educators as possible.

Equally troubling, the statement

> If you weigh the time spent learning the method that does not make sense to you, perhaps time favors my [alternative division] procedure.

expresses a paradoxical perspective that seems to have taken hold in part of the mathematics education community: if a piece of mathematics is hard to understand, excise it from the curriculum rather than investigate it carefully, with the cooperation of content experts whose mathematical perspective is both wide and deep, in order to find a clear, concise, and logical explanation.

Most students and educators will agree with the following assertion in an arithmetic textbook that was published four hundred years ago [H]://

> Division is esteemed one of the busiest operations of Arithmetick, and such as requireth a mynde not wandering, or settled upon other matters.

However, there is no mystery to the long division algorithm. It has been known, understood, and used to advantage for centuries. The only remaining mystery is why a clear explanation of that algorithm is absent from teacher training courses offered by American schools of education.

In conclusion, the reader is invited to share the following thought-provoking excerpt from an article by Thomas Groleau, a professor of business administration, who explains well the importance of formal mathematical competency, even for those students whose professional activities will be largely nonmathematical [Gr]:

> Martial arts training in the sixth century started with the Horse Stance: legs wide apart, knees bent, back straight. Students would hold this uncomfortable position for up to an hour and simply concentrate. However, the Horse Stance doesn't show up in combat. I've seen plenty of "Kung Fu" flicks, and never seen Chuck Norris use the Horse Stance in a fight.
>
> So what was the Horse Stance's value? It built stamina, strength, balance and concentration, all vitally important combat skills. Perhaps more important, it weeded out those unwilling or unable to pay the price to learn the secrets of the martial arts masters. While today's martial arts students don't face the Horse Stance[3], they still endure rigorous (and often tedious) foundational training.
>
> A traditional math curriculum can do the same for us. Algebra, geometry, calculus, etc. provide a skill foundation for OR/MS [Operations research/ management science]. Students who don't understand basic algebra tend to struggle with spreadsheets; the concept of absolute versus relative cell reference confuses them. On the other hand, students who understand traditional math tend to pick up spreadsheet skills with relative ease.

Groleau’s analogy is apt. In any rigorous university course, students who wish to understand symbol-laden texts and lectures, keep up with homework assignments, and also perform well on examinations must arrive prepared with the prerequisite algebraic skills that are the mathematical analogues of stamina, strength, balance, and concentration. An appropriate regimen of rigorous foundational training is a critical ingredient of any K-8 math curriculum designed to prepare as many students as possible for success in high school and college mathematics curricula that emphasize FAMC, as well as for their later pursuit of mathematics-based careers.

Appendix: Standard algorithms for multi-digit multiplication and division

This appendix describes algorithms and illustrates tableaux for multi-digit multiplication and division of decimal whole numbers, in a way that makes as clear as possible the inverse relationship between multiplication and division. Consider, for example, the multiplication problem 27*4572 = 123444, and its inverse problem, 123444 $\div$ 27 = 4572. At the simplest level, the inverse relationship between these problems is:

- 27*4572 is calculated by starting with zero and adding 27 4572 times to get up to 123444, whereas
- 123444 $\div$ 27 is calculated by starting with 123444, then subtracting 27 4572 times to get down to zero.

[3] Actually, they do; see [Y], for example. Groleau’s error only strengthens his argument. I am indebted to my colleague Ethan Akin for directing me to this reference.

Of course, the stated procedures are ridiculously inefficient. The decimal system of base-10 representation of whole numbers makes possible an enormous increase in efficiency, assuming a certain “library” of prerequisite skills. Formulating the contents of that library requires the following simple piece of terminology: a **place value numbe**r is the number obtained by writing a digit from 1 through 9, possibly followed by one or more zeroes.

The place value numbers were listed earlier, in the discussion of the division algorithm in Section 1, as:

1,2,3,4,5,6,7,8,9,
10,20,30,40,50,60,70,80,90,
100,200,300,400,500,600,700,800,900, , , and so on.

The required prerequisite skills for manipulating whole numbers expressed in base 10 notation are:

For multi-digit multiplication: finding the product of any number and a place value number, and finding the sum of two numbers.

For long division: finding the product of any number and a place value number, and finding the difference of two numbers.

Any whole number expressed in base-10 notation is the sum of place value numbers that correspond to its digits in an obvious way. For example, 4572 = 4000 + 500 + 70 + 2. It is this decomposition that lies at the heart of both the multiplication and division algorithms for such numbers.

Let’s begin with the multiplication algorithm, which is easy to summarize: expand the second factor as the sum of place value numbers and apply the distributive law, beginning with the smallest place value number. In the demonstration example, for example, calculate 27*4572 = 27*(4000 + 500 + 70 + 2) = 27*(2 + 70 + 500 + 4000) as the sum

```
27 * 2    =      54
27 * 70   =    1890
27 * 500  =   13500
27 * 4000 =  108000
Total:       123444
```

The multiplication tableau shown is a very slight variation of the following one, hereafter referred to as “standard,” in which trailing zeroes are omitted:

```
      27
 x  4572
      54
    189
   135
  108
  123444
```

The alert reader will observe that the method shown presumes knowledge of column addition and so does not adhere strictly to the stated prerequisite skill list for the multiplication algorithm. Of course, it is possible to add just two numbers at a time, and, despite the resulting compromise of efficiency, the above tableau will be adjusted in that manner in order to clarify the inverse relationship between the tableaux for multiplication and division. The result is the modified multiplication tableau:

```
Start:                0
Add 27 * 2     =     54
Subtotal:            54
Add 27 * 70    =   1890
Subtotal:          1944
Add 27 * 500   =  13500
Subtotal:         15444
Add 27 * 4000  = 108000
Total:         = 123444
```

Next consider the inverse division problem 123444 ÷ 27. The formal specification of the long division algorithm has already been given in Section 1. Informally, the procedure for the current problem consists of starting with the dividend 123444 and then successively subtracting multiples of the divisor 27 by place value numbers. At each stage, choose the largest possible place value number that makes the difference non-negative, and continue until that difference is less than the divisor. That difference is the remainder and the total number of 27's subtracted is the quotient.

In a first attempt to carry out the division algorithm, it is useful to list the multiples of 27 by place value numbers as in the following table. The second column lists multiples of 27 by the single-digit integers in the first column. To obtain each subsequent column, multiply the column to its left by 10.

k	27*k	*10	*100	*1000
1	27	270	2700	27000
2	54	540	5400	54000
3	81	810	8100	81000
4	108	1080	10800	108000
5	135	1350	13500	135000
6	162	1620	16200	162000
7	189	1890	18900	189000
8	216	2160	21600	216000
9	243	2430	24300	243000

In summary, the table lists multiples of 27 organized as: one to nine 27´s, or 10´s of 27´s, or 100´s of 27´s, or 1000´s of 27´s. It is easy to scan the table to find the largest possible multiple of 27 that can be subtracted at each stage of the division algorithm.

The table above need not and should not be used by students who have good multiplication and estimation skills. Those who have not yet acquired proficiency can develop their skills by constructing such tables.

The following division tableau for the stated problem includes step by step comments that are appropriate during initial stages of instruction:

```
To begin, start with   123444 (which is the dividend).
Subtract 27 * 4000 =   108000
                       ------
which leaves            15444
Subtract  27 * 500 =    13500
                        -----
which leaves             1944
Subtract  27 * 70  =     1890
                         ----
which leaves               54
Subtract  27 * 2   =       54
                           --
which leaves (finally)     0, the remainder, since 0 < 27.
```

Reading down, the number of 27's subtracted was 4000 + 500 + 70 + 2 = 4572. Therefore the quotient is 4572 with remainder 0.

The modern long division tableau condenses the one above by

- leaving out the words;
- "bringing down" digits from the dividend only as they are needed; and
- writing the answer horizontally, on top of the dividend, rather than vertically.

In the division problem illustrated, the remainder is 0, as will always be the case for a division problem that is the inverse of a multiplication problem involving whole numbers. The reader should have no difficulty generalizing to the case of nonzero remainder: e.g., writing down the corresponding tableaux for the division problem $123456 \div 27 = 4572$ with remainder 12, and for its inverse problem, namely, start with 12, then add the product 4572*27 to get back to 123456.

It is instructive to write the modern division tableau and the above "wordy" modified tableau side by side. It is likely that many students would benefit by listing the summands of the quotient vertically, even when they use the modern tableau, and so these are included in parentheses, to the right of the arrows below. Educators with perfect allegiance to the modern tableau may choose to wean their students from this minor variation.

Modified division tableau: Modern division tableau:

```
                                              4572
To begin, start with  123444          27/ 123444  In the quotient,
Subtract  27 *4000 =  108000    --->(4000) 108      write 4 (thousands).
leaving                15444                154     Bring down the 4.
Subtract  27 * 500 =   13500    ---> (500)  135     Write 5 (hundreds).
leaving                 1944                 194    Bring down the 4.
Subtract  27 *  70 =    1890    --->  (70)   189    Write 7 (tens).
leaving                   54                   54   Bring down the 4.
Subtract  27 *   2 =      54    --->   (2)     54   Write 2 (ones).
finally leaving            0                    0   is the remainder.
```

Writing the modified multiplication tableau obtained earlier for 4572 * 27 = 123444 next to the modified division tableau, above left, for 123444 ÷ 27 = 4572 reveals the underlying design principle of the latter. Simply stated, it is that, in their modified forms presented above,

The modern division tableau, read from top to bottom, is identical to the standard multiplication tableau, read from bottom to top.

Indeed, in the multiplication tableau below at the left, the product 123444 is built up from 0 by adding the place value numbers of the multiplier 4572, whereas in the division tableau to its right, subtracting those same place value numbers, but in reverse order, reduces the dividend 123444 back down to 0:

Multiply: 4572 * 27 = 123444

```
Start with               0
Add     27 * 2 =        54
Subtotal:               54
Add    27 * 70 =      1890
Subtotal:             1944
Add   27 * 500 =     13500
Subtotal:            15444
Add 27 * 4000 =     108000
Total:              123444
```

Divide: 123444 ÷ 27 = 4572

```
Start with                123444
Subtract  27*4000   =   108000
leaving                    15444
Subtract   27*500   =      13500
leaving                     1944
Subtract    27*70   =       1890
leaving                       54
Subtract     27*2   =         54
finally leaving                0
```

The inverse relationship between the two tableaux is perhaps easier to see when they are written horizontally, with intermediate sums and differences omitted:

Multiply: `0 + 2*27 + 70*27 + 500*27 + 4000*27 = 123444`
Divide: `123444 – 4000*27 – 500*27 – 70*27 – 2*27 = 0`

In the final analysis, it is the exact and beautiful symmetry between the reasonably obvious tableau for multiplication, and the much less obvious modern tableau for long division, that underlies the popularity and utility of the latter, beginning more than five centuries ago [C] and continuing until the present day.

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